Quantum Geometrodynamics for Black Holes and Wormholes

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Abstract.

The geometrodynamics of the spherical gravity with a selfgravitating thin dust shell as a source is constructed. The shell Hamiltonian constraint is derived and the corresponding Schroedinger equation is obtained. This equation appeared to be a finite differences equation. Its solutions are required to be analytic functions on the relevant Riemannian surface. The method of finding discrete spectra is suggested based on the analytic properties of the solutions. The large black hole approximation is considered and the discrete spectra for bound states of quantum black holes and wormholes are found. They depend on two quantum numbers and are, in fact, quasicontinuous.

1 Introduction

In the absence of the theory of quantum gravity we have to construct a new theory every time we want to quantize some classical gravitating object. The most interesting models of the kind are black holes and cosmological models . Here we are interested in the quantum black holes models. These models deserve consideration for many reasons. The black hole physics gives us an example of the strong gravitational fields. The existence of the event (apparent) horizons causes the Hawking's evaporation of the black holes . The fate of the evaporating black holes becomes a subject of interest. The quantum theory may throw some light to many problems of the classical black hole physics.

There were many many attempts to construct such quantum models. The most interesting of them are described in [6], [7], [8].In these papers the quantum theory of the eternal Schwarzschild black hole was constructed . The authors introduced many useful and important mathematical tools (in the present paper the canonical transformation found by Kuchar is widely used. but the physical result is rather trivial and obvious. Namely the quantum functional depends only on the Schwarzschild masses. The reason for this is that the eternal Schwarzschild black hole has no dynamical degrees of freedom. That is , all the matter collapsed classically and all possibly dynamical degrees of freedom died in the singularity. In the present paper we treat (or try to) the simplest quantum black hole model. "The simplest" means that we consider a spherically symmetric gravity with a self-gravitating thin dust shell as a source. We constructed the classical geometrodynamics for the system. Quantization of such a model leads to the Schroedinger equation in finite differences in the coordinate representation. The shift in the argument is along imaginary axes which has very important consequences. One of them is that the wave function which are the solution to such an equation should be analytical function on the appropriate Riemannian surface.

It should be noted that it is not the first time we are dialing with the finite differences equation . In the toy model constructed by one of us the Schroedinger equation in finite differences emerged as a result of the use of the proper time quantization. Unlike this toy model the present consideration deals with the canonical formalism from the very beginning. Thus , the results do not depend on the choice of time. And the appearance of the finite differences equation is due to the nonlocal nature of the corresponding

Hamiltonian operator.

In the ordinary quantum mechanics we are dealing with the second order differential equations. And we demand that the solution should be at least two times differentiable. To find eigenfunctions and spectrum we need to specify a class of functions, usually by imposing appropriate boundary conditions. In our case of the finite differences operator we must specify a class of function by demanding analyticity (except in the branching points). Our experience with the toy model shows that the boundary conditions help us to select the wave eigenfunctions (though up to the infinite degeneracy) but the are useless in finding the mass spectrum. To find the spectrum we need to know only the analytic properties of the solutions, namely their branching points.

It makes our life easier. We do not need to solve the Schroedinger equation. We should only to investigate the behavior of the solutions in the singular points of the corresponding equation.

The plan of the paper is the following. In the Section 2 we remind some facts from the classical dynamics of the thin dust shells. The classical geometrodynamics is developed in Section 3. Section 4 is devoted to the derivation of the shell's Hamiltonian constraints. The quantum geometrodynamics of the spherical gravity with the thin shell is considered in Section 5. Section 6 is dealing with the quasiclassical limit of our Schroedinger equation which in our case is the same as the large black holes regime. In this section we found the quantum black hole and wormhole discrete mass spectra.

2 Preliminaries.

The aim of this paper is to consider a geometrodynamics , both classical and quantum of the spherically symmetric gravitational field with self-gravitating dust thin shell as a source.

We start with the description of the model. This is just a self-gravitating spherically symmetric dust thin shell, endowed with a bare mass M. The whole space-time is divided into three different regions: the inner part (I), the outer part (II) containing no matter fields separated by thin layer III, containing the dust matter of the shell.

The general metric of a spherically symmetric spacetime has the form:

$$ds^2 = -N^2 dt^2 + L^2(dr + N^r dt)^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where (t, r, θ, ϕ) are space-time coordinates, N, N^r, L, R are some functions of t and r only. Trajectory of the thin shell is some 3-dimensional surface Σ in space-time given by some function $\hat{r}(t)$: $\Sigma^3 = \{(t, r, \theta, \phi) : r = \hat{r}(t)\}$. In region I $r < \hat{r} - \epsilon$, in a region II $r > \hat{r} + \epsilon$, region III is a thin layer $\hat{r} - \epsilon < r < \hat{r} + \epsilon$.

We require that metric coefficients N, N^r, L and R are continuous functions but jump discontinuities could appear in their derivatives at the points of Σ when the limit $\epsilon \to 0$ is taken.

Contrast to the flat space-time the normal vector to the surface $R = const$ may not be only spacelike but also timelike.In the first case the invariant

$$F = g^{\alpha\beta} R_{,\alpha} R_{,\beta} > 0 \tag{2}$$

and the corresponding region is called R-region (fig. 1) (here $g_{\alpha\beta}$ is a metric tensor, $g^{\alpha\beta}$ is its inverse, $R_{,\alpha}$ denotes the partial derivative with respect to the corresponding coordinate, Greek indices run from 0 to 3). In the flat case R-region occupies the whole space-time. In the second case

$$F < 0 \tag{3}$$

Such a region is called the T-region (the notions of R- and T- regions were introduced in [1]). It is easy to show that the condition $\dot{R} = 0$ (dot denotes time derivative) cannot be satisfied in a T-region, hence it should be either $\dot{R} > 0$ (this region of inevitable expansion is called T_+-region), or $\dot{R} < 0$

(inevitable contraction, a T_--region). Correspondingly, it is impossible to get $R' = 0$ (prime denotes the spatial derivative) in R-regions, and a region with $R' > 0$ is called an R_+-region, while that with $R' < 0$ is an R_- region. The R_+- and R_-- regions correspond to different sides of the Einstein-Rosen bridge (see fig. 1).

The solution of Einstein equations representing the Schwarzschild (spherically symmetric) black hole is well known and can be put in the form

$$ds^2 = -FdT^2 + F^{-1}dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4}$$

where

$$F = 1 - \frac{2Gm}{R} \tag{5}$$

and m is the total mass (energy) of the system, G is the gravitational constant (equal to the inverse square of Planckian mass, $G = M_{pl}^{-2}$,in the chosen units with $c = \hbar = 1$; note also that in these units and the radius has dimension of inverse mass).

The metric of 3-dimensional surface Σ^3 , representing the evolution of the thin shell can be written as

$$ds^2\Big|_\Sigma = -d\tau^2 + \hat{R}(\tau)(d\theta^2 + \sin^2\theta d\phi^2) \tag{6}$$

Here τ is the proper time of the observer sitting on the shell. From Einstein equations one obtains the equations of motion of the shell in the form

$$\sigma_{in}\sqrt{\dot{\hat{R}}^2 + F_{in}} - \sigma_{out}\sqrt{\dot{\hat{R}}^2 + F_{out}} = \frac{GM}{R} \tag{7}$$

(see [2]). The quantity σ has the following meaning, $\sigma = +1$ if radii R increase in the outward normal direction to the shell, and $\sigma = -1$ if radii decrease. Therefore in the R-regions σ does not change its sign, the latter being the sign of an R-region. It is clear now that on "our" side of Einstein-Rosen bridge we have $\sigma = +1$, this we shall call the "black hole case" , while on the "other" side $\sigma = -1$, this we shall call the "wormhole case". On fig. 2-5 we show all possible junctions of inner and outer regions with Schwarzschild masses m_{in} and m_{out} with thin shell of bare mass M. This figures are rather schematic. The space-time inside the shell (to the left of the trajectories) depends on whether there are other shells inside the given

one or not. To avoid such a concretization we draw the part of a complete inner Schwarzschild space-time with the left infinity. What important here are the junction of the space-times and the and the positions of the event horizons.

3 Canonical formalism for spherically symmetric gravity with thin shell.

The action functional for the system of spherically symmetric gravitational field and the thin shell is

$$S = S_{gr} + S_{shell} = \frac{1}{16\pi G} \int\limits_{I+II+III} {}^{(4)}R\sqrt{-g}d^4x + (surface\ terms) - \\ -M \int\limits_{\Sigma} d\tau \tag{8}$$

It consists of the standard Einstein-Hilbert action for the gravitational field and matter part of the action describes a thin shell of dust. The surface terms in the gravitational action and the falloff behavior of the metric and its derivatives were studied in details in [6]. So we will not consider this question and will use the results of Kuchar when needed. We will be interested in the behavior of the action and constraints on the surface Σ^3 representing the shell's trajectory.

The complete set of degrees of freedom of our system consists of the set of $N(r,t), N^r(r,t), L(r,t), R(r,t)$ which describe gravitational field and $\hat{r}(t)$ which describes the motion of the shell.

The metric (1) has the standard ADM form for 3+1 decomposition of a space-time with lapse function N , shift vector $N^i = (N^r, 0, 0)$ and space metric $h_{ik} = diag(L^2, R^2, R^2 \sin^2\theta)$ given foliation of the manifold on space and time. The scalar curvature density has the form

$$\begin{gathered} {}^{(4)}R\sqrt{-g} = \\ N\sqrt{h}\left({}^{(3)}R + ((TrK)^2 - TrK^2)\right) - \\ -2\left(\sqrt{h}K\right)_{,0} + 2\left(\sqrt{h}KN^i - \sqrt{h}h^{ij}N_{,j}\right)_{,i} \end{gathered} \tag{9}$$

where ${}^{(3)}R$ and K^{ij} are the scalar curvature of a space metric h_{ij} and exterior curvature tensor of a surface $t = const$. Substituting expression (1) for the metric into (9) we obtain the expression for internal and external curvatures of the surface $t = const$ in the form

$${}^{(3)}R = \frac{2}{R^2}\left(1 - \frac{(R')^2}{L^2} - \frac{2RR''}{L^2} + \frac{2RR'L'}{L^3}\right) \tag{10}$$

and

$$\begin{aligned} K^i_j &= diag(K^r_r, K^\theta_\theta, K^\phi_\phi) \\ K^r_r &= \frac{1}{NL}\left(\dot{L} - L'N^r - L(N^r)'\right), \\ K^\theta_\theta = K^\phi_\phi &= \frac{1}{NR}\left(\dot{R} - R'N^r\right) \end{aligned} \tag{11}$$

Here dot and prime denote differentiation in t and r respectively.

Contributions to the gravitational action from the terms containing total derivatives in (9) give rise to the surface terms which cancel each other at the common boundaries of regions I, II and II, III. So we are left with the surface terms at infinity which were extensively discussed in [6]. We will turn to them later.

The essential part of the action for gravitational field is just the ADM part of the action (8) with Lagrangian

$$L_{gr} = \frac{1}{16\pi G} NLR^2 \left((3)R - (TrK)^2 - TrK^2 \right) \tag{12}$$

Contribution to the action from the integral over the region III in the limit $\epsilon \to 0$ is only due to the term containing second derivative of R,namely

$$\int\limits_{III} \frac{1}{16\pi G} NLR^2 \ {}^{(3)}R = -\int\limits_{II} \frac{NRR''}{GL} = -\int\limits_{\Sigma} \frac{\hat{N}\hat{R}\,[R']}{G\hat{L}} \tag{13}$$

We will denote by hats variables on Σ and by $[\mathcal{A}] = \lim_{\epsilon\to 0}\left(\mathcal{A}(\hat{r}+\epsilon) - \mathcal{A}(\hat{r}-\epsilon)\right)$ a jump of variable $\mathcal{A}(r)$ on the shell surface.

Substituting the expression (1) into the shell part of the action we have:

$$S_{shell} = -M \int\limits_{\Sigma} \sqrt{\hat{N}^2 - \hat{L}^2\left(\hat{N}^r + \dot{\hat{r}}\right)^2} dt \tag{14}$$

The explicit form of the action (8) with metric (1) becomes

$$\begin{aligned} S &= \frac{1}{G} \int\limits_{I+II+III} \left(N\frac{L}{2}\frac{(R')^2}{2L} - \left(\frac{RR'}{L}\right)' + \frac{R}{N}\left(\dot{R} - R'N^r\right)\left((LN^r)' - \dot{L}\right) \right. \\ &+ \left. \frac{L}{2N}\left(\dot{R} - R'N^r\right)^2 \right) - \int\limits_{\Sigma} \left(\frac{\hat{N}\hat{R}\,[R']}{\hat{L}} - m\sqrt{\hat{N}^2 - \hat{L}^2\left(\hat{N}^r + \dot{\hat{r}}\right)^2} \right) dt \end{aligned} \tag{15}$$

The canonical formalism for this action can be described in the following way. Momenta conjugate to corresponding dynamical variables are

$$
\begin{array}{rcl}
P_N &=& \delta S\,/\,\delta \dot{N} = 0; \\
P_{N^r} &=& \delta S\,/\,\delta \dot{N}^r = 0 \\
P_L &=& \delta S\,/\,\delta \dot{L} = \frac{R}{GN}\left(R'N^r - \dot{R}\right) \\
P_R &=& \delta S\,/\,\delta \dot{R} = \frac{L}{GN}\left(R'N^r - \dot{R}\right) + \frac{R}{GN}\left((LN^r)' - \dot{L}\right) \\
P_{\hat{R}} &=& \delta S\,/\,\delta \dot{\hat{R}} = 0 \\
P_{\hat{L}} &=& \delta S\,/\,\delta \dot{\hat{L}} = 0 \\
\hat{\pi} &=& \delta S\,/\,\delta \dot{\hat{r}} = \dfrac{m\hat{L}^2(N^r + \dot{\hat{r}})}{\sqrt{\hat{N}^2 - \hat{L}^2(N^r + \dot{\hat{r}})}}
\end{array}
\tag{16}
$$

The action (15) rewritten in the Hamiltonian form becomes

$$
\begin{array}{c}
S = \int\limits_{I+II} \left(P_L\dot{L} + P_R\dot{R} - NH - N^r H_r\right) dr dt + \int\limits_{\Sigma} \hat{\pi}\dot{\hat{r}} - \\
\hat{N}\left(\hat{R}\,[R']\,/(G\hat{L}) + \sqrt{m^2 + \hat{\pi}^2/\hat{L}^2}\right) - \\
\hat{N}^r\left(-\hat{L}\,[P_L] - \hat{\pi}\right) dt
\end{array}
\tag{17}
$$

with

$$
\begin{array}{rcl}
H &=& G\left(\dfrac{LP_L^2}{2R^2} - \dfrac{P_L P_R}{R}\right) + \dfrac{1}{G}\left(-\dfrac{L}{2} - \dfrac{(R')^2}{2L} + \left(\dfrac{RR'}{L}\right)'\right) \\
H_r &=& P_R R' - LP_L'.
\end{array}
\tag{18}
$$

where $N, N^r, \hat{N}$ and $\hat{N}^r$ are Lagrange multipliers in the Hamiltonian formalism. The system of constraints contain two surface constraints in addition to usual Hamiltonian and momentum constraints of the ADM formalism.

ADM constraints:

$$
\left\{
\begin{array}{l}
H = 0 \\
H_r = 0
\end{array}
\right.
\tag{19}
$$

Shell constraints:

$$
\left\{
\begin{array}{l}
\hat{H}_r = \hat{\pi} + \hat{L}\,[P_L] = 0 \\
\hat{H} = \dfrac{R\,[R']}{GL} + \sqrt{M^2 + \hat{\pi}^2/\,L^2} = 0
\end{array}
\right.
\tag{20}
$$

4 Kuchar variables.

In the paper [6] Kuchar proposed canonical transformation of the variables (R, P_R, L, P_L) to new canonical set $(R, \bar{P}_R, M, P_M)$ in which Hamiltonian and momentum constraints given by (18) are equivalent to the very simple set of constraints :

$$\begin{array}{c} \bar{P}_R = 0 \\ M' = 0 \end{array} \tag{21}$$

The idea is to use the Schwarzschild anzatz for the space-time metric (4) instead of (1):

$$ds^2 = -F(R,m)dT^2 + \frac{1}{F(R,m)}dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{22}$$

where T, R and m are some functions of (r,t) and $F(R,m) = 1 - 2Gm/\, R$, and m, in general, is a function of r, m=m(r). Equating the two forms of the metric (1) and (22) we obtain the transformation between the two sets of dynamical variables.

The explicit form of the transformation is

$$\begin{array}{rcl} L &=& \sqrt{\frac{(R')^2}{F} - FP_m^2} \\ P_L &=& \frac{RFP_m}{G}\sqrt{\frac{(R')^2}{F} - FP_m^2} \\ R &=& R \\ \bar{P}_R &=& P_R + \frac{P_m}{2G} + \frac{FP_m}{2G} + \frac{(RFP_m)'RR' - RFP_m(RR')'}{GRF\left((R')^2\,/\,F - FP_m^2\right)} \end{array} \tag{23}$$

where $P_m = -T'$.

The Liouville form

$$\Theta = \int P_R\dot{R} + P_L\dot{L} \tag{24}$$

can be expressed in the new variables as follows:

$$\begin{array}{rcl} \Theta &=& \int P_m\dot{m} + \bar{P}_R\dot{R} + \frac{\partial}{\partial t}\left(LP_L + \frac{1}{2G}RR'\ln\left|\frac{RR' - LP_LG}{RR' + LP_LG}\right|\right) \\ &+& \frac{\partial}{\partial r}\left(\frac{1}{2G}R\dot{R}\ln\left|\frac{RR' + LP_LG}{RR' - LP_LG}\right|\right). \end{array} \tag{25}$$

When there is no shell the total derivatives in (25) give rise to some surface terms at infinities. As shown by Kuchar [6] the appropriate falloff conditions at infinities make the last surface term from (25) zero. Then it follows from (25) that $(R, \bar{P}_R, m, P_m)$ form a canonical set of variables and equation (23) describes a canonical transformation between (R, P_R, L, P_L) and $(R, \bar{P}_R, m, P_m)$.

5 Canonical variables and Hamiltonian constraint on the shell.

5.1 Shell variables.

In the presence of the thin shell the situation is different. Surface terms now should not be neglected.

Let us do the transformation (23) in regions I and II of our space-time.The Liouville form of our Hamiltonian system (17) has the form

$$\tilde{\Theta} = \int\limits_{I+II} P_R \dot{R} + P_L \dot{L} + \int\limits_{\Sigma} \hat{\pi}\dot{\hat{r}}. \tag{26}$$

After integration the total derivatives in (25) give some contribution to the Liouville form on Σ:

$$\begin{aligned}
\tilde{\Theta} &= \int\limits_{I+II} \bar{P}_R \dot{R} + P_m \dot{m} + \int\limits_{\Sigma} \left[L P_L + \frac{1}{2G} R R' \ln\left| \frac{RR' - LP_L G}{RR' + LP_L G} \right| \right] \dot{\hat{r}} dt \\
&- \int\limits_{\Sigma} \left[\frac{1}{2G} R \dot{R} \ln\left| \frac{RR' + LP_L G}{RR' - LP_L G} \right| \right] + \int\limits_{\Sigma} \hat{\pi}\dot{\hat{r}} \\
&= \int\limits_{I+II} P_m \dot{m} + \bar{P}_R \dot{R} + \int\limits_{\Sigma} \hat{p}\dot{\hat{r}} + \int\limits_{\Sigma} \hat{P}_{\hat{R}} \dot{\hat{R}}
\end{aligned} \tag{27}$$

where we denoted

$$\begin{aligned}
\hat{p} &= \hat{\pi} + L\,[P_L] \\
\hat{P}_{\hat{R}} &= \left[\frac{1}{2G} R \ln\left| \frac{RR' - GLP_L}{RR' + GLP_L} \right| \right]
\end{aligned} \tag{28}$$

and made use of the identity

$$\dot{\hat{R}} = \frac{d}{dt} R(t, \hat{r}(t)) = (\dot{R}(t,r) + R'(t,r)\dot{\hat{r}}(t))\;|_{r=\hat{r}(t)} \tag{29}$$

We see that this canonical transformation involves all the set of coordinates in the phase space $\Pi = \{(R(r,t), P_R(r,t), L(r,t), P_L(r,t), \hat{r}(t), \hat{\pi}(t))\}$ according to the formulae (23) and (28). Moreover it introduces additional pair of canonically conjugate variables $(\hat{R}, \hat{P}_{\hat{R}})$ on the shell.

In both inner and outer regions I and II constraints are simplified due to the canonical transformation as it was in the absence of the shell (21). The surface momentum constraint $\hat{H}_r = 0$ (18) takes the form

$$\hat{p} = 0 \tag{30}$$

5.2 Shell constraint. Special case.

In order to investigate the form of Hamiltonian surface constraint in new variables let us choose coordinates (r, t) in a specific way. If coordinate lines $t = const$ cross the shell perpendicularly then

$$\dot{\hat{r}} = -\hat{N}^r \tag{31}$$

This means according to (16) that

$$\hat{\pi} = 0 \tag{32}$$

Then the surface momentum constraint gives

$$GL\,[P_L] = R\,[FP_M] = 0 \tag{33}$$

Let us denote

$$\gamma = FP_M/\,L \tag{34}$$

We have from (28)

$$\hat{P}_{\hat{R}} = \frac{1}{2G} R \left[\ln \left| \frac{R' - \gamma}{R' + \gamma} \right| \right] = \frac{R}{G} \ln \left(\left| \frac{R'_{in} + \gamma}{R'_{out} + \gamma} \right| \sqrt{\left| \frac{F_{out}}{F_{in}} \right|} \right) \tag{35}$$

From (23 we get

$$\frac{R'}{L} = \sigma \sqrt{F + \gamma^2} \tag{36}$$

where $\sigma = \pm 1$ is the sign function taking its values according to whether radii increase in the outward normal direction to the shell ($\sigma = +1$) or they decrease ($\sigma = -1$). Using (35) and (36) we could find the expression for γ. Then from (36) we could determine the jump of R' across the shell surface as a function of $\hat{P}_{\hat{R}}, \hat{R}, m_{in}$ and m_{out}. Substituting this into the surface Hamiltonian constraint (20) we obtain the following expression

$$\hat{H} = \frac{\sigma_{in} R}{G} \sqrt{\sqrt{F_{out}} - \sqrt{F_{in}} \exp\left(\frac{G\hat{P}_{\hat{R}}}{R}\right)} \sqrt{\sqrt{F_{out}} - \sqrt{F_{in}} \exp\left(-\frac{G\hat{P}_{\hat{R}}}{R}\right)} - M = 0 \tag{37}$$

5.3 Shell constraint. General case.

In the previous subsection we managed to derive the shell constraint by finding so some continuous variable, namely γ (Eqn.34).Now we will try to do the same trick in general case $\pi \neq 0$. Let's consider the full time derivative of the shell radius.

$$\dot{\hat{R}} \equiv \frac{d}{dt}\hat{R}\left(r\left(t\right),t\right) = \dot{R} + R'\dot{r} \tag{38}$$

Using the definition $P_L = -\frac{R}{GN}\left(\dot{R} - R'N^r\right)$ we get:

$$\dot{\hat{R}} = -\frac{GNP_L}{R} + R'\left(\dot{r} + N^r\right) \tag{39}$$

Remembering that

$$\pi \equiv \frac{ML^2\left(N^r + \dot{r}\right)}{\sqrt{N^2 - L^2\left(N^r + \dot{r}\right)}}$$

we can find

$$L\left(N^r + \dot{r}\right) = \frac{\pi N}{L\sqrt{M^2 + \frac{\pi^2}{L^2}}}$$

Eqn.(39) now reads

$$\frac{\dot{\hat{R}}}{N} = -\frac{P_L}{R} + \frac{R'}{L}\frac{\pi}{L\sqrt{M^2 + \frac{\pi^2}{L^2}}} \tag{40}$$

It is now easy to see that the jump of $\dot{\hat{R}}$ across the shell is a linear combination of constraints

$$\left[\dot{\hat{R}}\right] = \frac{GN}{R}\left(\chi H - \frac{H^r}{L}\right) \tag{41}$$

where

$$\chi = \frac{\pi}{L\sqrt{M^2 + \frac{\pi^2}{L^2}}}$$

To go further we need Eqn.(**??**) which we now rewrite as follows

$$\beta \equiv e^{\frac{G\hat{P}_R}{R}} = \frac{\left(\frac{\dot{\hat{R}}}{N} + G\frac{P_L}{R}\left(1+\chi\right)\right)_{in}}{\left(\frac{\dot{\hat{R}}}{N} + G\frac{P_L}{R}\left(1+\chi\right)\right)_{out}} \equiv \frac{\alpha + y_{in}\left(1+\chi\right)}{\alpha + y_{out}\left(1+\chi\right)} \tag{42}$$

where $\alpha = \frac{\dot{\hat{R}}}{N}$ and $y = G\frac{P_L}{R}$.
The next step is to find the relation between α and $\dot{\hat{R}}$. From the definitions of P_L and $\dot{\hat{R}}$ we have

$$\frac{R'^2}{L^2} = F + y^2. = \frac{\alpha}{\chi} + \frac{y}{\chi}$$

. Solving this for y we get

$$y = \frac{\alpha \pm \sqrt{\alpha^2 - (\chi^2 - 1)(F\chi^2 - \alpha^2)}}{\chi^2 - 1} \tag{43}$$

Substituting this expression back into Eqn. 42 we obtain after simple algebraic transformations

$$\beta = \frac{z - \sigma_{in}\sqrt{z^2 + F_{in}}}{z - \sigma_{out}\sqrt{z^2 + F_{out}}} \tag{44}$$

$$where \quad z^2 = \frac{\alpha^2}{1 - \chi^2} \tag{45}$$

Making use of the momentum constraint we can wright the jump of y as follows

$$\sigma_{out}\chi\sqrt{\alpha^2 + (1 - \chi^2)F_{out}} = \sigma_{in}\chi\sqrt{\alpha^2 + (1 - \chi^2)F_{in}} - G\tfrac{M}{R}\chi\sqrt{1 - \chi^2} \tag{46}$$

Note that this is just the Eqn. (7) if we choose proper time (substituting $\dot{\hat{R}}^2$ for z^2.)
Squaring this equation we get

$$\begin{cases} \sigma_{in}\sqrt{z^2 + F_{in}} = -\frac{R[F]}{2MG} + \frac{MG}{2R} \\ \\ \sigma_{out}\sqrt{z^2 + F_{out}} = -\frac{R[F]}{2MG} - \frac{MG}{2R} \end{cases} \tag{47}$$

$$z = \pm\sqrt{\left(\frac{R[F]}{2MG}\right)^2 - \frac{1}{2}(F_{out} + F_{in}) + \frac{M^2G^2}{4R^2}} \tag{48}$$

Finally we get for the shell constraint

$$\beta - \frac{z + \dfrac{R\,[F]}{2MG} - \dfrac{MG}{2R}}{z + \dfrac{R\,[F]}{2MG} + \dfrac{MG}{2R}} = 0 \tag{49}$$

This constraint can be rewritten in a more elegant way

$$\begin{aligned}\sqrt{F_{out}}\left(\pm\sqrt{\left(\tfrac{[m]}{M}\right)^2 - 1 + G\tfrac{m_{out}+m_{in}}{R} + \tfrac{M^2G^2}{4R^2}} - \tfrac{[m]}{M} + \tfrac{MG}{2R}\right)e^{\frac{G\hat{P}_R}{2R}} - \\ \sqrt{F_{in}}\left(\pm\sqrt{\left(\tfrac{[m]}{M}\right)^2 - 1 + G\tfrac{m_{out}+m_{in}}{R} + \tfrac{M^2G^2}{4R^2}} - \tfrac{[m]}{M} - \tfrac{MG}{2R}\right)e^{-\frac{G\hat{P}_R}{2R}} = 0\end{aligned} \tag{50}$$

where the Schwarzschild anzatz was substituted for F's .

It can be easily shown that the above constraint is equivalent to that one derived in the previous subsection. And this proves that the Hamiltonian constraint (37) is valid for nonzero values of π as well. We see that the only remained classical constraint of the shell Hamiltonian dynamics can be written in various equivalent forms (e.g. , Eqns (37), (49) or (50). But, of course, quantum mechanically all these forms are no more equivalent. So , we need some criteria to choose among them. One of this criteria well be considered in the last section. In what follows we will consider the following squared version of the Hamiltonian constraint (37) as the suitable classical counterpart for the quantum constraint for the wave function Ψ

$$C = F_{out} + F_{in} - \sqrt{F_{out}}\sqrt{F_{in}}\left(\exp\frac{G\dot{P}_R}{R} + \exp-\frac{G\dot{P}_R}{R}\right) - \frac{M^2G^2}{R^2} \tag{51}$$

The Hamiltonian constraint (37) was derived under the assumption that both F_{in} and F_{out} are positive . It is possible, of course to derive analogous constraints in $T_{\pm}$-regions, where $F < 0$. But, instead, we make the following substitution

$$\sqrt{F} \to F^{1/2} \tag{52}$$

and consider this function as a function of complex variable. Then the point of the horizon $F = 0$ becomes a branching point , and we need the rules of

the bypass. We assume the following

$$
\begin{array}{rl}
F^{1/2} & = |F|\, e^{i\phi} \\
\phi = 0 & \text{in } R_{+}\text{-region} \\
\phi = \pi/2 & \text{in } T_{-}\text{-region} \\
\phi = \pi & \text{in } R_{-}\text{-region} \\
\phi = -\pi/2 & \text{in } T_{+}\text{-region}
\end{array}
\tag{53}
$$

for the black hole case , and

$$
\begin{array}{rl}
\phi = \pi & \text{in } R_{+}\text{-region} \\
\phi = -\pi/2 & \text{in } T_{-}\text{-region} \\
\phi = 0 & \text{in } R_{-}\text{-region} \\
\phi = \pi/2 & \text{in } T_{+}\text{-region}
\end{array}
\tag{54}
$$

for the wormhole case. The reason for such analytical continuation is that we are able to get the single equation on the wave function Ψ which covers all four patches of the complete Penrose diagram for the Schwarzschild spacetime.Some consequences of this fact will become evident in section 6.

6 Quantized spherical gravity with thin shell.

We now turn to the Dirac constraint quantization procedure .

It is convenient to make a canonical transformation from $(\hat{R}, \hat{P}_{\hat{R}})$ to $(\hat{S}, \hat{P}_S)$:

$$\left\{ \begin{array}{rcl} \hat{S} & = & \dfrac{\hat{R}^2}{(2GM)^2} = \dfrac{\hat{R}^2}{R_g^2} \\ \hat{P}_S & = & R_g^2 \dfrac{\hat{P}_{\hat{R}}}{2\hat{R}} \end{array} \right. \tag{55}$$

where R_g is the gravitational radius of the shell. Dimensionless variable $\hat{S}$ is the surface area of the shell measured in the units of horizon area of the shell of mass M.

The phase space of our model consist of coordinates $(R(r), \tilde{P}_R(r), m(r), P_m(r), \hat{S}, \hat{P}_S, \hat{r}, \hat{p}_r)$ $r \in (-\infty, \hat{r} - \epsilon) \bigcup (\hat{r} + \epsilon, \infty)$. Then the wave function in coordinate representation depends on configuration space coordinates:

$$\Psi = \Psi(R(r), m(r), \hat{S}, \hat{r}) \tag{56}$$

and all the momenta become operators of the form

$$\begin{array}{cc} \tilde{P}_R(r) = -i\, \delta/\, \delta R(r) & P_m(r) = -i\, \delta/\, \delta m(r) \\ \hat{P}_S = -i\, \partial/\, \partial \hat{S} & \hat{p}_r = -i\, \partial/\, \partial \hat{r} \end{array} \tag{57}$$

ADM and shell constraints (19) and (20) become operator equations on Ψ. The set of ADM constraints is equivalent to the set of constraints (21) in Kuchar variables which could be easily solved on quantum level. Indeed, in the regions I and II the equations

$$\left\{ \begin{array}{rcl} \partial\Psi/\, \partial R(r) & = & 0 \\ M'(r)\Psi & = & 0 \end{array} \right. \tag{58}$$

express the fact that wave function does not depend on $R(r)$ and the dependence on $M(r)$ is reduced in each region I and II to $\Psi \equiv \delta(M - M_\pm)$ where $M_\pm$ defined in the regions I (-) and II (+) do not depend on r. They equal to Schwarzschild masses in the inner and outer regions M_{in} and M_{out} in (50).

The set of shell constraints (20) impose further restrictions on Ψ. First of them takes the form

$$\partial\Psi/\, \partial\hat{r} = 0 \tag{59}$$

in new variables according to (30). So the only nontrivial equation is the shell constraint (51) (or (37,(49),(50)which are classically equivalent to the Eqn.(51))

$$\hat{C}\left(m_+, m_-, \hat{S}, -i\hbar\partial/\partial\hat{S}\right) = 0 \qquad (60)$$
$$\Psi = \Psi(m_+, m_-, \hat{S})$$

The operator $\hat{C}$ contains the exponent of the of the momentum $\hat{P}_S$. This exponent becomes an operator of finite displacement when $\hat{P}_S$ becomes differential operator:

$$e^{\frac{G\hat{P}_R}{R}} = e^{\frac{\hat{P}_S}{2GM^2}}\Psi = e^{-i\frac{m_p^2 l}{M^2}\frac{\partial}{\partial\hat{S}}}\Psi = \Psi(m_+, m_-, \hat{S} - \xi i) \qquad (61)$$

where m_{pl} is Plank mass and $\xi = \frac{1}{2}(\frac{m_{pl}}{M})^2$

The constraint C is nonlinear in $\hat{S}$ and P_S so the question of ordering should be solved when replacing dynamical variables by operators. It is proposed in [3]to choose the symmetric ordering

$$A(\hat{S})B(\hat{P}_S) \to 1/2\left\{A(\hat{S})B(-i\hbar\partial\,/\,\partial\hat{S}) + \overline{B}(i\hbar\partial\,/\,\partial\hat{S})\overline{A}(\hat{S})\right\} \qquad (62)$$

where A and B are some functions of $\hat{S}$ and $\hat{P}_S$ respectively and $\overline{A}$ denotes complex conjugation.

With operator ordering (62) we must add to the constraint (51) the complex conjugate part .

The constraint $\hat{C}$ becomes an equation in finite differences if we express $\hat{R}$ through $\hat{S}$ and substitute the expression (**??**) to the differential operator. Finally we get

$$\begin{aligned} F_{out}^{1/2}F_{in}^{1/2}\left(\Psi(s+i\xi)+\Psi(s-i\xi)\right)+\overline{F_{out}^{1/2}F_{in}^{1/2}}(s+i\xi)\Psi(s+i\xi)+ \\ \overline{F_{out}^{1/2}F_{in}^{1/2}}(s-i\xi)\Psi(s-i\xi) = 2(F_{out}+F_{in}-\frac{1}{4s})\Psi(s) \end{aligned} \qquad (63)$$

We have mentioned already that the classically equivalent constraints give inequivalent quantum theories. This is well known fact. We suggest that the criterion to choose the correct quantum theory is the behavior of the wave functions in the quasiclassical regime.In our case this means the large black holes limit. Indeed , the parameter $\zeta = \frac{1}{2}(\frac{m_{pl}}{m})^2$ becomes small for large

masses , and the expansion with respect to this parameter is equivalent to the expansion in Planckian constant $\hbar$ ($m_{pl} = \sqrt{\hbar/Gc}$). In the next Section we will consider this quasiclassical limit and show that our choice for the quantum constraint is a good one (at least in the case of one thin shell). At the and of this Section we would like to make an important remark. Our quantum equation 63 (which is just a Schroedinger equation) is the equation in finite differences rather than differential equation, and the shift in argument is along an imaginary axis. In the case of differential equation we we require the solution to be differentiable sufficiently many times . Similarly, we have to demand the solutions of our finite differences equation 63 to be analytical functions. This condition is very restrictive but unavoidable. Our previous experience (see [2]) shows that it is the analyticity of the wave functions and not the boundary conditions that lead to the existence of the discrete mass (energy) spectrum for bound states. How it works in the quasiclassical regime we will see in the next Section.

7 Large black holes.

The finite difference equation (63) becomes an ordinary differential equation in quasiclassical limit which is the same as the limit of large ($m \gg m_{pl}$) black holes. Indeed the parameter of finite displacement of the argument of Ψ in (63) $\xi = (m_{pl} \,/\, M)$ becomes small and we could cut the Tailor expansion

$$\Psi(\hat{S} + \xi i) = \Psi(\hat{S}) + i\xi\Psi'(\hat{S}) - \frac{\xi^2}{2}\Psi''(\hat{S}) + ... \tag{64}$$

at the second order.

Now we will analyze the behavior of the solutions of equation (63) in this quasiclassical limit at singular points.

We will restrict the consideration to the case of flat inner region $m_- = 0$, so we denote $m_+ = m$. It is convenient to redefine the area variable $\hat{S}$ so that

$$\begin{cases} S &= \dfrac{\hat{R}^2}{(2Gm)^2} = \dfrac{\hat{R}^2}{\tilde{R}_g^2} \\ P_S &= \tilde{R}_g^2 \dfrac{\hat{P}_{\hat{R}}}{2\hat{R}} \end{cases} \tag{65}$$

area is now measures in the units of horizon area of a black hole with Schwarzschild mass M ($\tilde{R}_g$ is its gravitational radius). In these units the displacement parameter

$$\zeta = \frac{m_{pl}}{m} \tag{66}$$

and the equation (63) reads as

$$\begin{aligned} e^{i\phi}\sqrt{|F|}\,(\Psi(s+i\zeta) + \Psi(s-i\zeta)) + e^{-i\phi}\sqrt{|F|}(s+i\zeta)\Psi(s+i\zeta) + \\ e^{-i\phi}\sqrt{|F|}(s-i\zeta)\Psi(s-i\zeta) = 2(F - \tfrac{1}{4s})\Psi(s) \end{aligned} \tag{67}$$

where ϕ is the phase of $F^{1/2}$. It should be chosen in different R- and T-regions according to the arguments of section 5. In the last formula we must take the Tailor expansion on ξ up to second order.

$$\begin{aligned} \Psi|_{S\pm\zeta i} &\approx \Psi(S) \pm \Psi'(S)\zeta i - \frac{\zeta^2}{2}\Psi''(S)\dots \\ F^{\frac{1}{2}}\Big|_{S\pm\zeta i} &= \sqrt{1 - \frac{1}{\sqrt{s \pm \zeta i}}} \approx F^{\frac{1}{2}}\left(1 \pm \frac{1}{2FS^{3/2}}\zeta i + \left(\frac{3}{8FS^{5/2}} + \frac{1}{8F^2S^3}\right)\zeta^2\right)\dots \end{aligned} \tag{68}$$

This leads to ordinary differential equations of second order, which are different in R_+, R_-, T_+ and T_- regions due to the different values of the phases in Eqn.(67). The interesting for us singular points of these differential equations are

$$S = \infty \text{ and } S = 1. \tag{69}$$

In the quasiclassical limit our requirement of the analyticity of the solutions to the exact equation (67) transforms into the requirement that the branching points of the leading terms in the solutions to the approximate equations should be of the same kind . Thus, we need to keep only those terms in the corresponding equations that give us these leading terms. Below we consider the black hole case only. The results are easily translated to the wormhole case.

The singular point $S = \infty$ in the region R_+ lies in a classically forbidden region as far as we restrict ourselves with bound motions of the shell only. In order to analyze the behavior of Ψ in this region we should take (67) with $\phi = 0$ and expand all the quantities in terms of y, where $s = (1+y)^2$ The result is

$$\Psi_{yy} - \frac{1}{y}\Psi_y + \frac{1}{\zeta^2}\left(1 - \frac{M^2}{m^2} + \frac{1}{2y}(2 - \frac{M^2}{m^2})\right)\Psi = 0 \tag{70}$$

The leading term of the solution is

$$\begin{aligned} \Psi &\sim y^{\frac{1}{2} - \frac{\frac{M^2}{m^2} - 2}{4\mu\zeta^2}} \exp(-\mu y), \\ \mu &= \frac{1}{\zeta}\sqrt{\frac{M^2}{m^2} - 1},\ y \gg \zeta \end{aligned} \tag{71}$$

For another singular point in R_+ region , that is for $S \to 1 + 0$ we have ($s = (1+z^2)^2$)

$$\Psi_{zz} - 3z\Psi_z + \frac{16z}{\zeta^2}\left(1 - \frac{M^2}{4m^2}\right)\Psi = 0 \tag{72}$$

with leading term

$$\Psi \sim 1 - \frac{8}{3\zeta^2}\left(1 - \frac{M^2}{4m^2}\right) y^{3/2} \tag{73}$$

$$y = \sqrt{z},\ s \gg \zeta,\ y \gg \zeta,\ \ \zeta \ll 1$$

Comparing the types of the branching points at $s \to \infty$ and $s \to 1+0$ we can conclude that

$$\frac{2 - \frac{M^2}{m^2}}{4\zeta\sqrt{\frac{M^2}{m^2} - 1}} = n, \quad n = integer \tag{74}$$

This is the first quantization condition. We will not consider here the wormhole case . Note only that , as can be shown , the positive values of quantum number n correspond to black holes while negative n correspond to wormholes.

In the T_--region (which classically is a region of inevitable contraction) , i.e., for $s \to 1-0$ ($s = (1+y)^2,\ y < 0,\ \zeta \ll |y| \ll 1$) have first order differential equation (due to complex conjugation introduced earlier the leading terms containing second derivatives of the wave function cancel each other) with following leading term in the solution

$$\Psi \sim \exp\left(i\frac{8}{3\zeta^2}\left(1 - \frac{M^2}{4m^2}\right)(-y)^{3/2}\right) \tag{75}$$

which is just the ingoing wave as it should be expected for the quasiclassical limit in the region of the inevitable contraction.That is why we have chosen the of the function $(F)^{1/2} = e^{i\phi}|F|^{1/2}$ in the T_-region (Which classically the region of inevitable expansion) the choice of $\phi = -\frac{\pi}{2}$ leads to the outgoing wave as a solution. Note also that to the our requirement of the analyticity the solution (leading term) in the T_-region should be the analytical continuation of the solution in the R_+region. And we see that this is indeed the case.

We do not consider here separately the asymptotics in R_- region near the horizon ($s \to 1+0$) because it differs from the corresponding solution in R_+-region only by the sign in front of the second term.

Let us now turn to the asymptotics of the solutions in R_--region for $s \to \infty$. Due to the minus sign in front of $F^{1/2}$ the equation for the wave function in a R_--region is quite different from that in a R_+-region

$$\Psi_{yy} - \frac{1}{y}\Psi_y - \frac{1}{\zeta^2}\left(16y^2 + 1 - \frac{M^2}{m^2})\right)\Psi = 0 \tag{76}$$

The leading term of the asymptotic is now the following

$$\Psi \sim y^{\frac{\frac{M^2}{m^2}-1}{8\zeta}} \exp -(\frac{2}{\zeta}y^2) \tag{77}$$

Note that the falloff in the R_--region is much faster than it is in the R_+-region. This is a quite reasonable result because it means that the quantum shell in the black hole case can penetrate into the R_--region (which is completely forbidden for the classical motion) but the probability of such an event is negligible small.

And , again, comparing the types of the branching points at $s \to 1+0$ and $s \to \infty$ in the R_--region we get

$$\frac{\frac{M^2}{m^2}-1}{8\zeta} = \frac{1}{2} + p, \qquad p = positive \quad integer \tag{78}$$

The appearance of the4 second quantum number is rather surprising but it allows some explanation. We discuss this point in the last Section.

Combining (74) and (78) we get

$$\frac{(\frac{M^2}{m^2}-1)^{3/2}}{2-\frac{M^2}{m^2}} = \frac{1+2p}{n} \tag{79}$$

and

$$m = \frac{\sqrt{2}\sqrt{1+2p}}{\sqrt{\frac{M^2}{m^2}-1}} m_{pl} \tag{80}$$

For $p \gg |n|$ we have

$$m \approx 2\sqrt{p} \quad m_{pl} \tag{81}$$

This corresponds to the shells of large mass whose mean value radius is rather close to the horizon. In the opposite case , $p \ll n$,

$$m \approx \sqrt{2}(1+2p)^{1/6} n^{1/3} m_{pl} \tag{82}$$

This corresponds to the massive shells with mean radius very far from horizon.

This the end of this Section we would like to consider the behavior of the solutions in the vicinity of the horizon (sub-Planckian deviation), where $|y| \gg \zeta$ ($s \sim 1$). To be specific we will be interested in the solutions R_+ and T_- regions. The expansion (68) is no more valid for the function $F^{1/2}(s \pm i\zeta)$ but it is still valid for the wave Ψ. Keeping the leading terms only we have now

$$\Psi_{ss}(s) - \frac{2}{\zeta}\Psi_s(s) + \left(\frac{4}{\alpha\zeta^{5/2}}(1 - \frac{M^2}{4m^2}) - \frac{2}{\zeta^2}\right)\Psi(s) = 0 \qquad (83)$$

with the solution

$$\Psi \sim e^{ks}, \quad k \approx -\frac{1}{\zeta} \pm \sqrt{-\frac{4}{\alpha\zeta^{5/2}}(1 - \frac{M^2}{4m^2})} \qquad (84)$$

The coefficient α equals to 1 in the R_+-region and to imaginary unit i in the T_--region.

In the R_+ region

$$k \approx -\frac{1}{\zeta} \pm i\sqrt{-\frac{4}{\zeta^{5/2}}(1 - \frac{M^2}{4m^2})} \qquad (85)$$

and we have superposition of two waves (ingoing and out outgoing) with relatively equal amplitudes.

in the T_--region

$$k \approx -\frac{1}{\zeta} \pm \sqrt{-\frac{4i}{\zeta^{5/2}}(1 - \frac{M^2}{4m^2})} = -\frac{1}{\zeta} \pm \frac{\sqrt{2}}{\zeta^{5/4}}\sqrt{(1 - \frac{M^2}{4m^2})}(1+i) = \qquad (86)$$

$$\left(\pm\frac{\sqrt{2}}{\zeta^{5/4}}\sqrt{(1 - \frac{M^2}{4m^2})} - \frac{1}{\zeta})\right) \pm i\frac{\sqrt{2}}{\zeta^{5/4}}\sqrt{(1 - \frac{M^2}{4m^2})}$$

The existence of two waves in the T_--region reflects the quantum trembling of the horizon . But the outgoing wave is enormously damped relative to the ingoing wave (of course , in the T_-- region the situation is exactly inverse one). It is this damping that cause (in a quasiclassical regime) existence of the single ingoing wave in the T_--region at the distances larger than Planckian.

8 Discussion

In the concluding remarks we would like to discuss the obtained results.

(1) We constructed the geometrodynamics for the spherical gravity with self-gravitating thin dust shell as a source. Such a shell provides us with the only dynamical degree of freedom which is otherwise absent in pure spherical gravity. We managed to separate the field canonical variables describing the gravitational field outside the shell from the single pair of the shell canonical variables which are just the radius of the shell and the corresponding conjugate momentum. The Hamiltonian constraint is derived for the shell which depends only on the invariants of the inner and outer parts of the manifold and on the parameters of the shell. The quantum functional is subject to both quantum ADM (field) constraints and the shell constraint. After solving the (trivial) quantum field constraints we are left with the functional which depends on the inner and outer masses of the corresponding Schwarzschild manifolds and is function of the radius of the shell. Thus , the functional becomes a wave function , and the remaining shell constraint is just a Schroedinger equation for this wave function.

(2) The obtained Schroedinger equation is the equation is the equation in the finite differences rather than differential equation. And the shift is along an imaginary axis . Dealing with the differential equations we always require (or assume) that the solutions should be sufficiently differentiable. Analogously in our situation we must require solutions to be analytic functions except some isolated points . Our equation has branching points , so , the solutions will have branching points as well.

(3) The Schroedinger equation we obtained contains , as a coefficient function , the square root of the $F = 1 - \frac{2Gm}{R}$, which is invariant function of the Schwarzschild solution. This function is positive outside the event horizons on the both sides of the Einstein-Rosen bridge (R_+ and R_-- regions) and it is negative beyond the horizons in the T_+-region of the inevitable expansion and in the T_--region of the inevitable contraction. We suggest to consider the square root as a function of complex variable ,$(F)^{1/2}$ acquiring different phases in patches of the complete Schwarzschild manifold. The aim of such a procedure is twofold. First, it allows us to obtain a common wave function covering the hole Penrose diagram ($R_\pm$ and $T_\pm$-regions). And , second, we remove the double cover degeneracy when the same value of radius R corresponds to two different points , on in the $R_+(T_+)$-region and

the other is in the $R_-(T_-)$ -region. Now this different points with the same value of radius lie in different sheets of the Riemannian surface.

(4)The requirement of the analyticity of the wave function on the corresponding Riemannian surface means that the branching points should be of the same kind in order to be connected by cuts. In another words , the number of the Riemannian sheets emerging at the branching points must be the same.

In our case we have the branching points at infinity and at the horizon. But , we have two different horizons. One of them separating, say ,R_+ and T_--regions , lies on the one sheet and the other one , separating T_- and R_+-regions lies on another sheet of the Riemannian surface . Thus comparing the branching points at infinity and at the horizon in R_+-region we obtain the first quantum number characterizing the mass spectrum of the system in question and, comparing these points in the T_- region we obtain the second quantum number. Thus , the mass spectrum of the black holes and wormholes should depend on two quantum numbers . (Note that using the same method one can obtain the famous spectra like oscillator, hydrogen atom and so on).

(5)It is well known that the classical theory may give rise to different inequivalent quantum theories. The origin of this phenomenon is a non-commutativity of dynamical variables and their conjugate momenta. The investigation of the quasiclassical limit helps to choose the "correct" quantum version . In our case the quasiclassical regime coincide with the limit of large (comparing to the Planckian mass) black holes. The finite differences equation can now be expanded in series with respect to the small parameter and we can cut the series to obtain the differential equation. We showed that our choice of the quantum Hamiltonian give a good quasi-classics (ingoing wave in the T_-region , and outgoing wave in the T_+-region, as should be expected). Moreover we showed that the black hole and wormhole discrete mass spectra is determined by two quantum number making this spectra quasicontinuous. This resembles the appearing of the fine structure due to removing some degeneracy (in our case it is a double covering degeneracy). In the ordinary quantum mechanics we are used to the fact that the number of quantum numbers equals to the numbers of degrees of freedom. From the first sight we have only one dynamical degree of freedom in our problem. Indeed, the motion of the spherically symmetric thin shell is described by the radius as a function of time in Lagrangian picture and by the radius and its

conjugated momenta in the Hamiltonian picture. But actually we have two different parts of the Penrose diagram with the same value of radius , namely R_+ and R_--regions (and T_+ and T_--regions). And by our consideration they lie on different sheets of the Riemannian surface. Of course, classical motion is forbidden in R_-region in the black hole case (in R_+region in the wormhole case). But in the quantum theory such motion is allowed. Thus , we have in fact two degrees of freedom . And this is just the origin if the second quantum number.

(6)This last remark concerns the problem of small black holes with mass about Planckian mass or smaller. The shift in our equation in finite differences is of order of the horizon size or even larger. This means that for small masses our equation does not feel the very existence of the horizons. And it gives some hope that there are no black holes at all with masses smaller than the Planckian mass.

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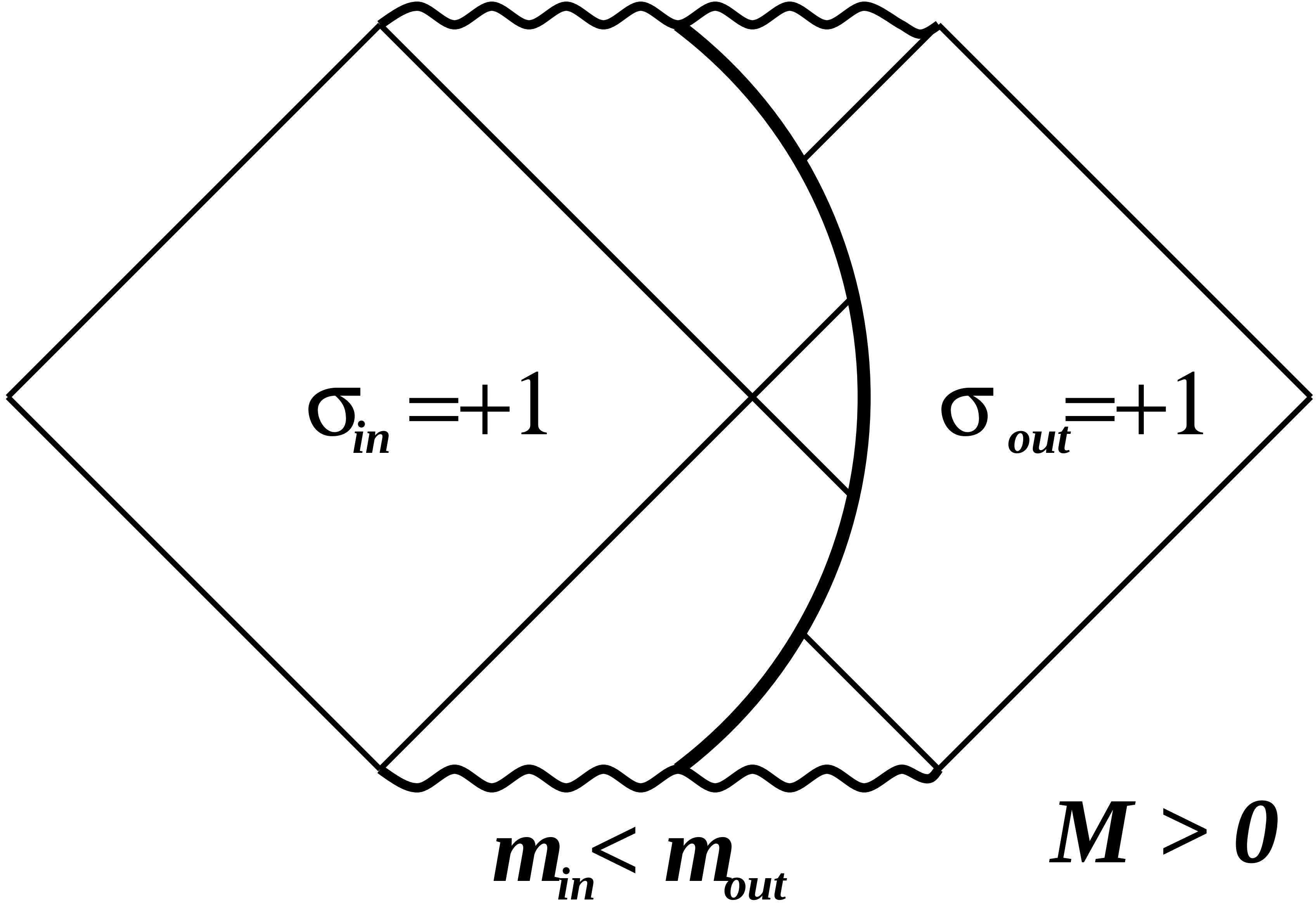

Fig. 2. Black hole case.

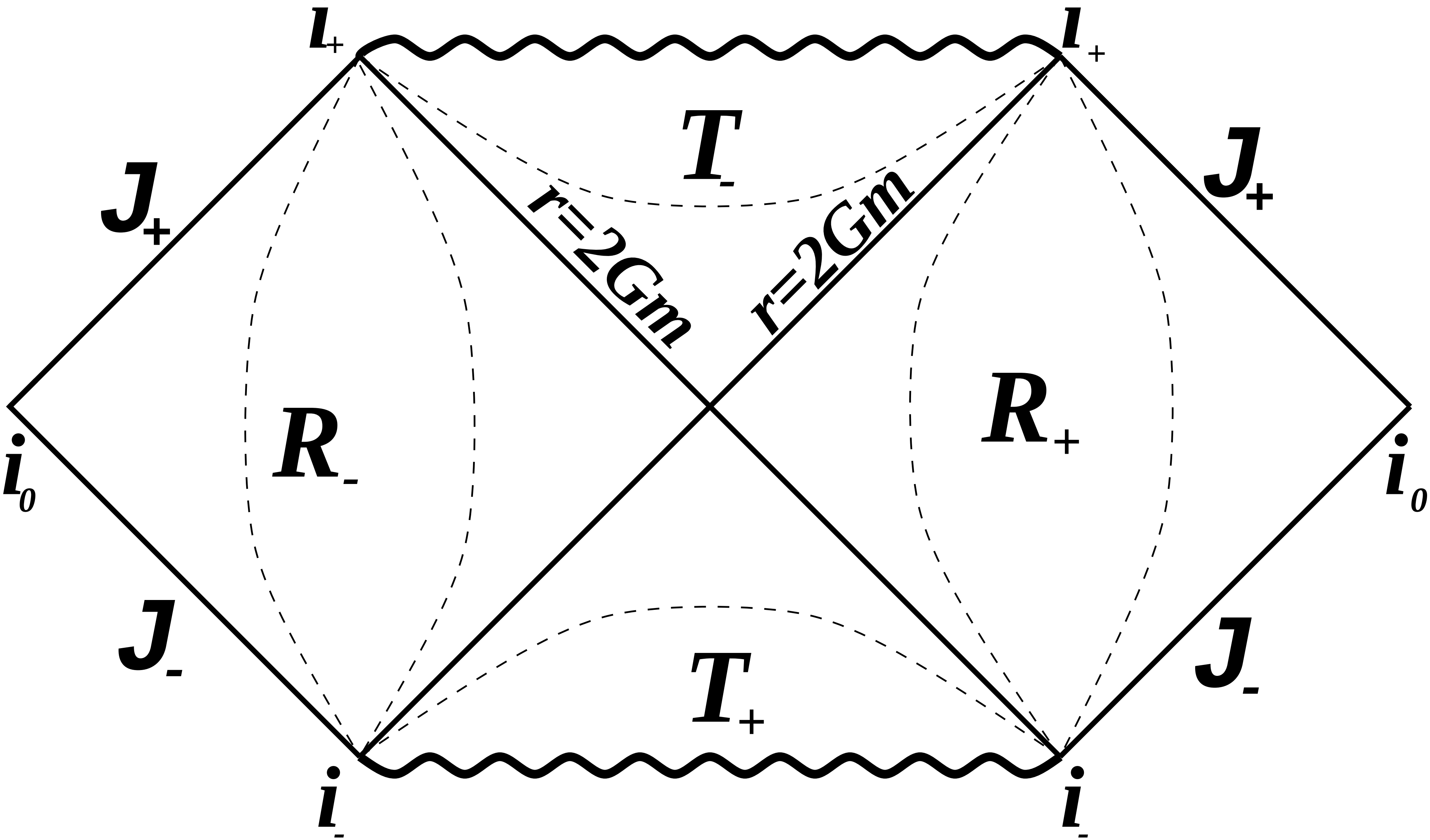

Fig. 1. Penrose diagram for Schwarzchild black hole.
Dashed lines are curves of constant radius.

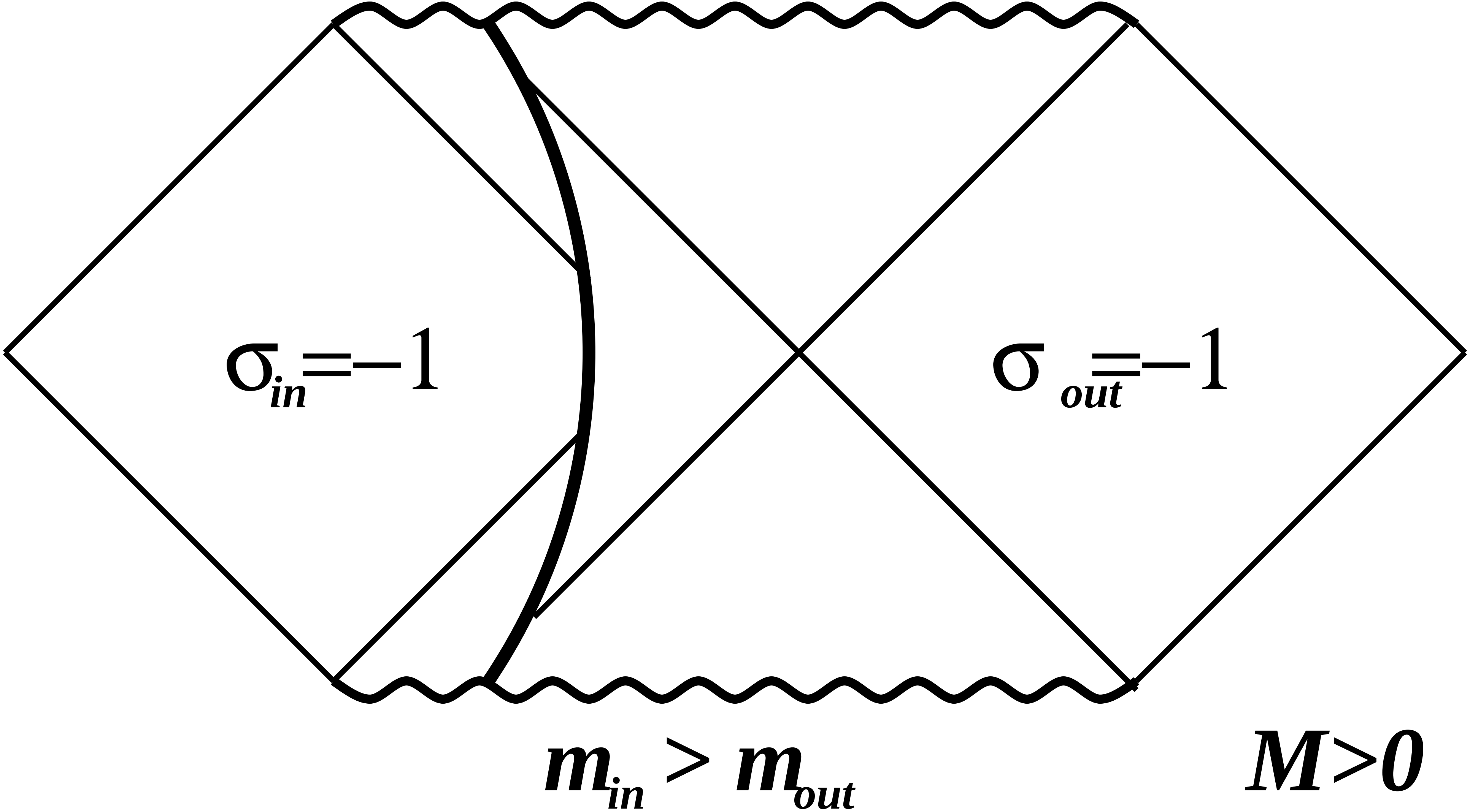

Fig. 3. Black hole case. In- and out- regions are interchanged.

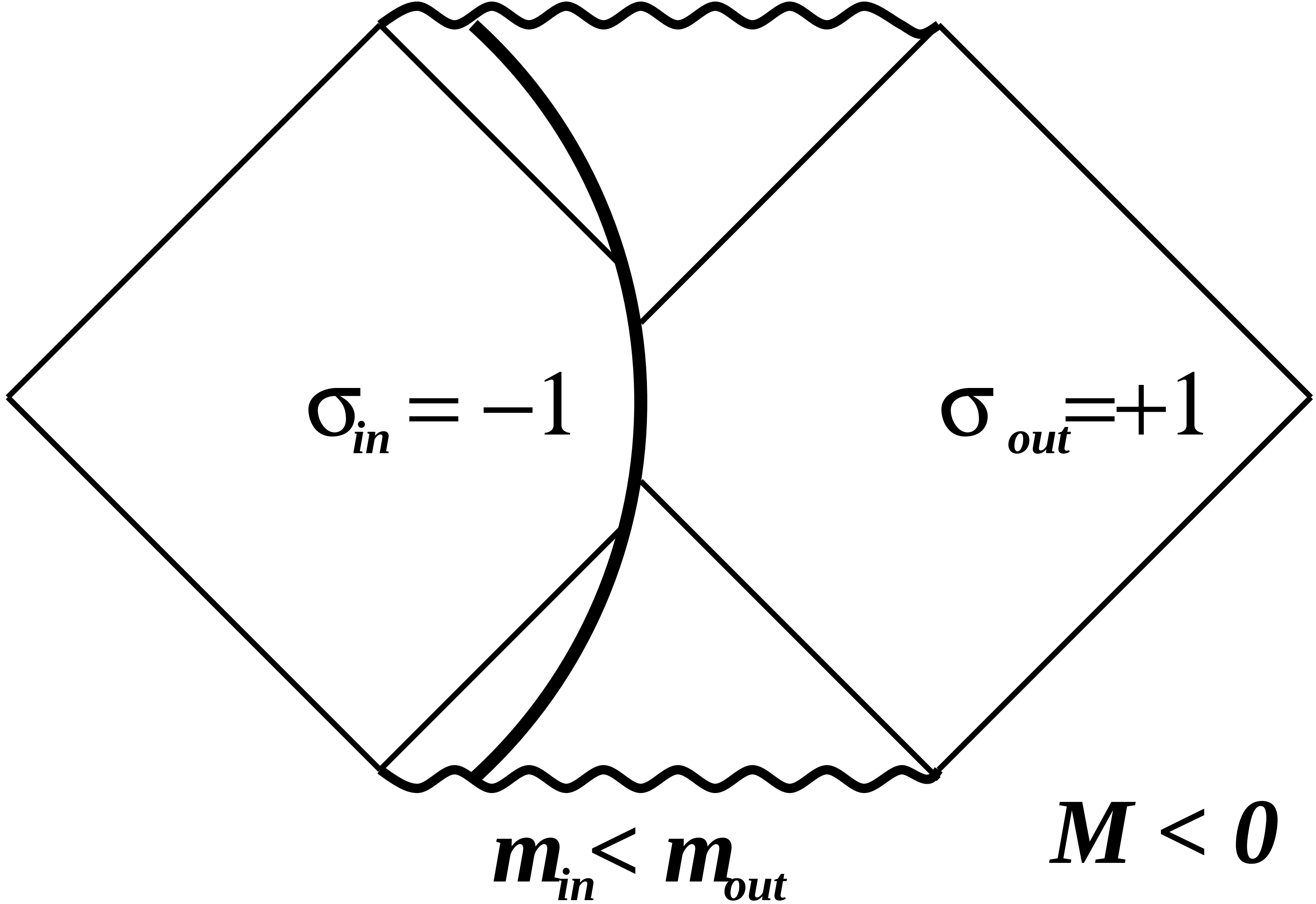

Fig. 5. Unphysical.

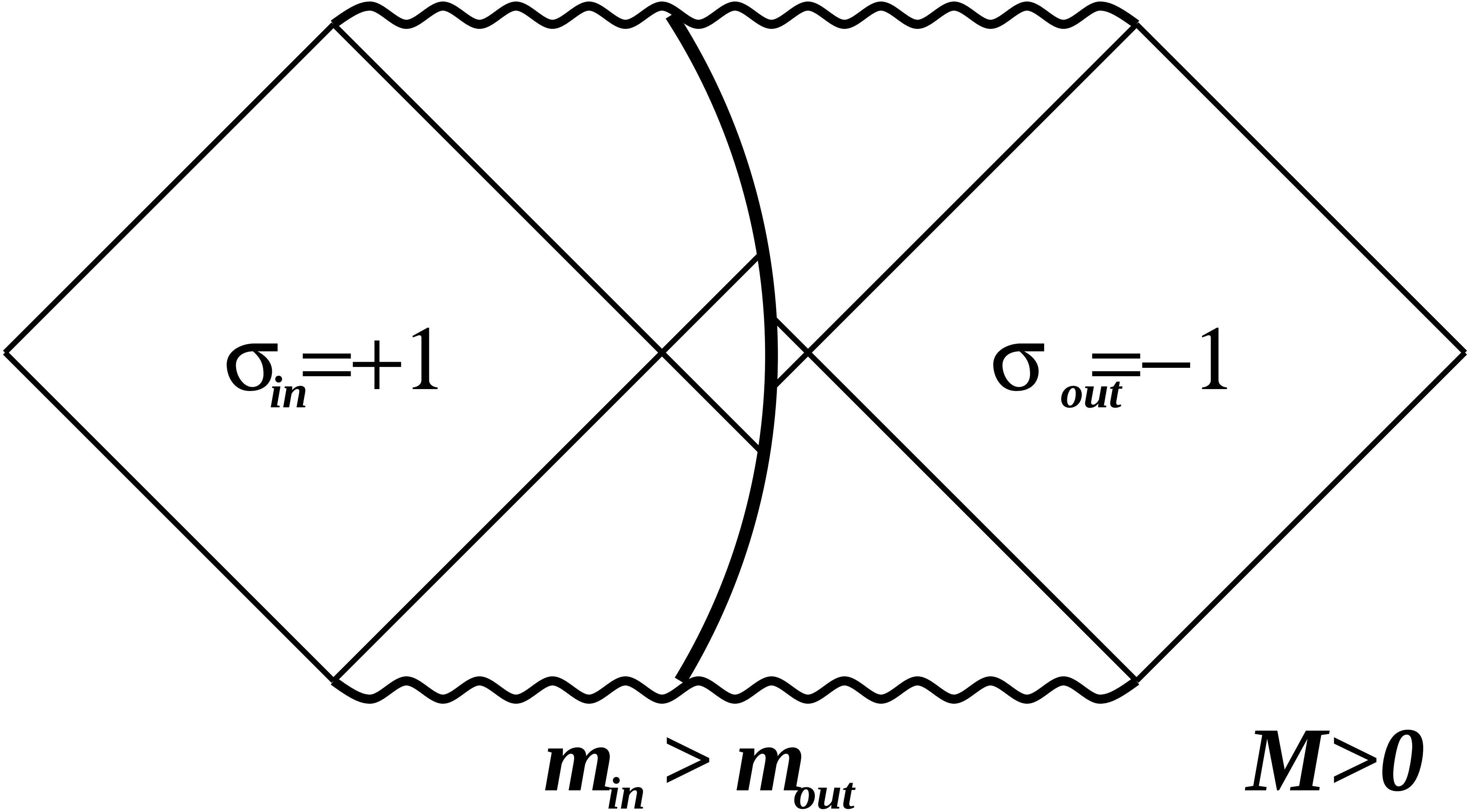

Fig. 4. Wormhole case.